Math 150, ***Lecture Notes- Bonds*** **Name______________**

Section 3.3 *Increasing and Decreasing Functions and the First Derivative Test*

Increasing and Decreasing Functions

In this section you will learn how derivatives can be used to *classify* relative extrema as either relative minima or relative maxima. First, it is important to define increasing and decreasing functions.

Definitions of Increasing and Decreasing Functions

A function f is **increasing** on an interval if for any two numbers x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) < f(x_2)$.

A function f is **decreasing** on an interval if for any two numbers x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) > f(x_2)$.

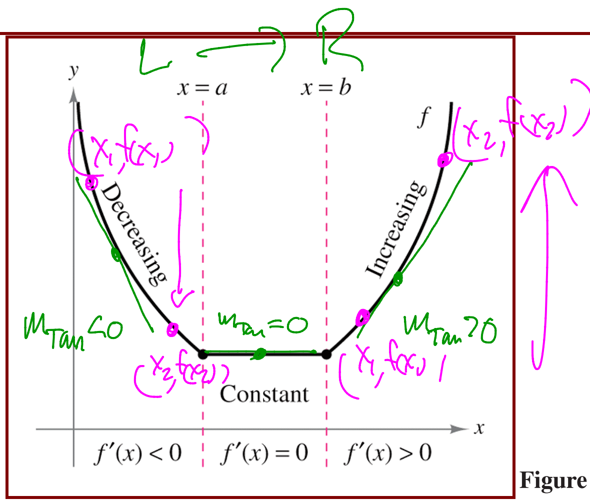


Figure 3.15

A function is increasing if, *as x moves to the right*, its graph moves up, and is decreasing if its graph moves down. For example, the function in Figure 3.15 is decreasing on the interval $(-\infty, a)$, is constant on the interval (a, b), and is increasing on the interval (b, ∞). As shown in Theorem 3.5 below, a positive derivative implies that the function is increasing; a negative derivative implies that the function is decreasing; and a zero derivative on an entire interval implies that the function is constant on that interval.

THEOREM 3.5 Test for Increasing and Decreasing Functions

Let f be a function that is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b).

1. If $f'(x) > 0$ for all x in (a, b), then f is increasing on $[a, b]$.
2. If $f'(x) < 0$ for all x in (a, b), then f is decreasing on $[a, b]$.
3. If $f'(x) = 0$ for all x in (a, b), then f is constant on $[a, b]$.

NOTE The conclusions in the first two cases of Theorem 3.5 are valid even if $f'(x) = 0$ at a finite number of x-values in (a, b).

Ex.1 Use calculus to find the open intervals on which $f(x)=x^3-\frac{3}{2}x^2$ is increasing or decreasing.

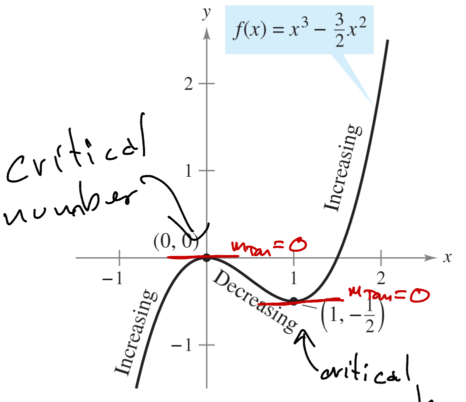


Figure 3.16

f' +++ | − − − | +++

$x=-2$ 0 $x=\frac{1}{2}$ 1 $x=2$

$$\frac{d}{dx}[f(x)] = \frac{d}{dx}\left[x^3-\frac{3}{2}x^2\right]$$

$$f'(x) = 3x^2 - \frac{3}{2}\cdot 2x$$

$$f'(x) = 3x^2-3x$$

Find critical numbers:

(A) $f'(x)=0$

$3x^2-3x=0$

$3x(x-1)=0$

either

$3x=0$, or $x-1=0$

$x=0$, or $x=1$

or (B) $f'(x)$ is undefined

f' is a polynomial & f' is never undefined

Test $f'(x)$:

$f'(-1) = 3(-1)[(-1)-1]$

$= -3[-2]$

$= 6 > 0$

Chart will be on test!

$f'(\frac{1}{2}) = 3(\frac{1}{2})[(\frac{1}{2})-1]$

$= \frac{3}{2}[-\frac{1}{2}]$

$= -\frac{3}{4} < 0$

$f'(2) = 3(2)[(2)-1]$

$= 6(1)$

$= 6 > 0$

L → R

Interval	$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
Test Value	$x=-1$	$x=\frac{1}{2}$	$x=2$
Sign of $f'(x)$	$f'(-1) > 0$ (+)	$f'(\frac{1}{2}) < 0$ (−)	$f'(2) > 0$ (+)
Conclusion	f is increasing	f is decreasing	f is increasing

So, f is increasing on the intervals $(-\infty, 0)$ and $(1, \infty)$ and decreasing on the interval $(0, 1)$, as shown in Figure 3.16.

Guidelines for Finding Intervals on Which a Function Is Increasing or Decreasing

Let f be continuous on the interval (a, b). To find the open intervals on which f is increasing or decreasing, use the following steps.

find f'

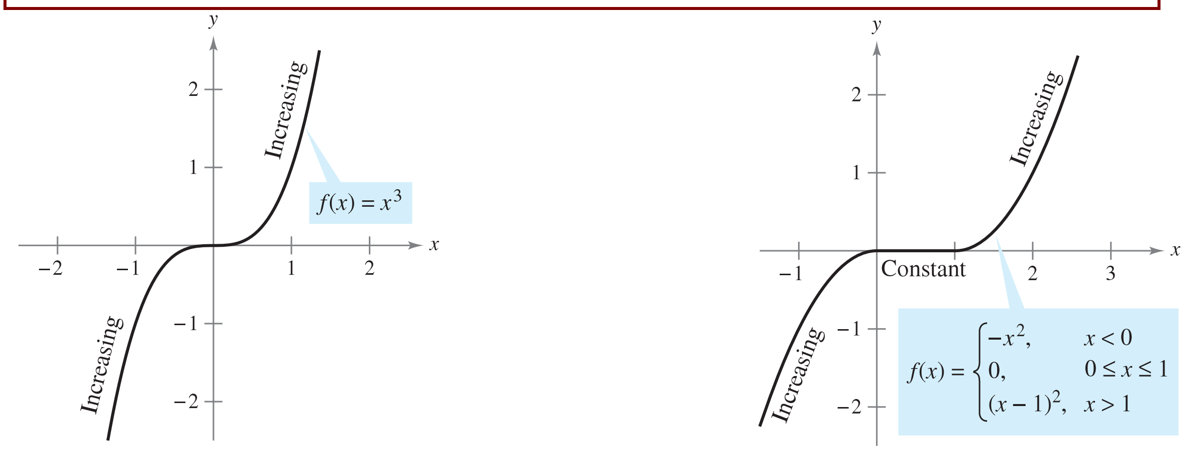
(A) $f' = 0$ or (B) f' is undefined

1. Locate the critical numbers of f in (a, b), and use these numbers to determine test intervals.

choose Test values & Evaluate f'

2. Determine the sign of $f'(x)$ at one test value in each of the intervals.
3. Use Theorem 3.5 to determine whether f is increasing or decreasing on each interval.

These guidelines are also valid if the interval (a, b) is replaced by an interval of the form $(-\infty, b)$, (a, ∞), or $(-\infty, \infty)$.



(a) Strictly monotonic function

(b) Not strictly monotonic

Figure 3.17

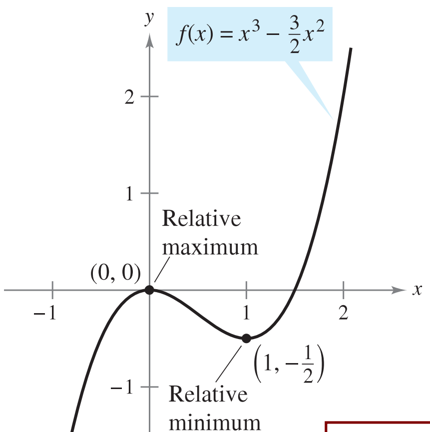
A function is **strictly monotonic** on an interval if it is either increasing on the entire interval or decreasing on the entire interval. For instance, the function $f(x) = x^3$ is strictly monotonic on the entire real number line because it is increasing on the entire real number line, as shown in Figure 3.17(a). The function shown in Figure 3.17(b) is not strictly monotonic on the entire real number line because it is constant on the interval $[0, 1]$.

The First Derivative Test

After you have determined the intervals on which a function is increasing or decreasing, it is not difficult to locate the relative extrema of the function. For instance, in Figure 3.18 (from Example 1), the function

$$f(x) = x^3 - \frac{3}{2}x^2$$

has a relative maximum at the point $(0, 0)$ because f is increasing immediately to the left of $x = 0$ and decreasing immediately to the right of $x = 0$. Similarly, f has a relative minimum at the point $\left(1, -\frac{1}{2}\right)$ because f is decreasing immediately to the left of $x = 1$ and increasing immediately to the right of $x = 1$. The following theorem, called the First Derivative Test, makes this more explicit.



Relative extrema of f
Figure 3.18

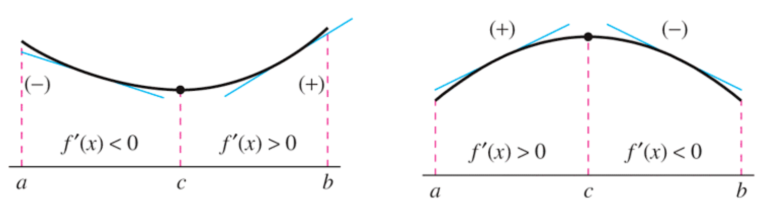
THEOREM 3.6 The First Derivative Test

Let c be a critical number of a function f that is continuous on an open interval I containing c. If f is differentiable on the interval, except possibly at c, then $f(c)$ can be classified as follows.

1. If $f'(x)$ changes from negative to positive at c, then f has a *relative minimum* at $(c, f(c))$.

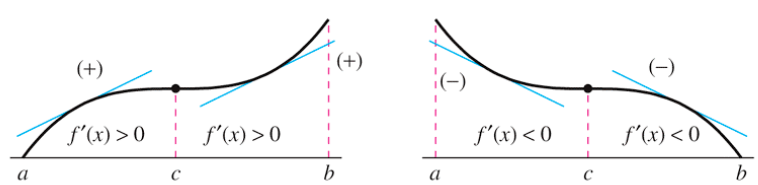
2. If $f'(x)$ changes from positive to negative at c, then f has a *relative maximum* at $(c, f(c))$.

3. If $f'(x)$ is positive on both sides of c or negative on both sides of c, then $f(c)$ is neither a relative minimum nor a relative maximum.



Relative minimum

Relative maximum



Neither relative minimum nor relative maximum

Ex.2 Applying the First Derivative Test

Find the relative extrema of $f(x)=\frac{1}{2}x-\sin(x)$ in the interval $(0,2\pi)$.

$\frac{d}{dx}[f(x)] = \frac{d}{dx}[\frac{1}{2}x - \sin(x)]$

$f'(x) = \frac{1}{2} - \cos(x)$ ✓

Find critical numbers:

(A) $f'(x) = 0$

$\frac{1}{2} - \cos(x) = 0$

$\frac{1}{2} = \cos(x)$

$x = \frac{\pi}{3}$, or $x = \frac{5\pi}{3}$

(B) f' is undefined

Never

Test $f'(x)$

$f'(\frac{\pi}{6}) = \frac{1}{2} - \cos(\frac{\pi}{6})$

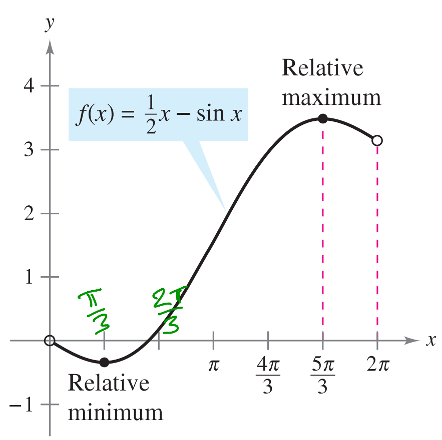
$= \frac{1}{2} - \frac{\sqrt{3}}{2} = -$

$f'(\pi) = \frac{1}{2} - \cos(\pi)$

$= \frac{1}{2} - (-1) = +$

$f'(\frac{11\pi}{6}) = \frac{1}{2} - \frac{\sqrt{3}}{2} = -$

Interval	$(0, \frac{\pi}{3})$	$(\frac{\pi}{3}, \frac{5\pi}{3})$	$(\frac{5\pi}{3}, 2\pi)$
Test Value	$x = \frac{\pi}{6}$	$x = \pi$	$x = \frac{11\pi}{6}$
Sign of $f'(x)$	$f'(\frac{\pi}{6}) < 0$ $-$	$f'(\pi) > 0$ $+$	$f'(\frac{11\pi}{6}) < 0$ $-$
Conclusion	f is decreasing	f is increasing	f is decreasing

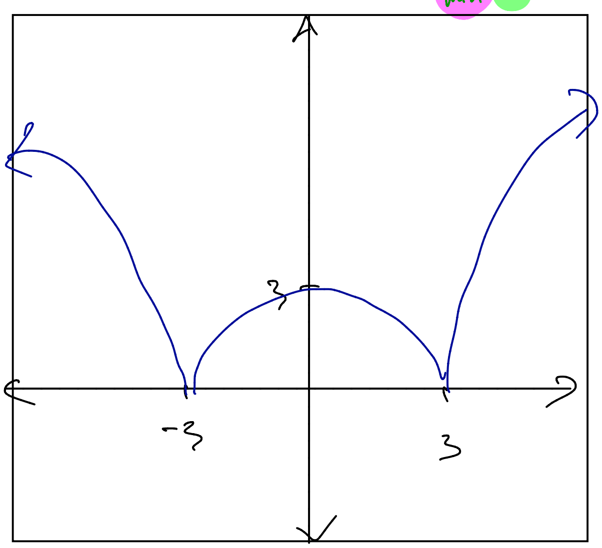


A relative minimum occurs where f changes from decreasing to increasing, and a relative maximum occurs where f changes from increasing to decreasing.

Ex.3 Find the relative extrema of $f(x)=(x^2-9)^{\frac{2}{3}}$.

Interval	$(-\infty,-3)$	$(-3,0)$	$(0,3)$	$(3,\infty)$
Test Value	$x=-5$	$x=-1$	$x=1$	$x=5$
Sign of $f'(x)$	$f'(-5)<0$ $\ominus$	$f'(-1)>0$ $\oplus$	$f'(1)<0$ $\ominus$	$f'(5)>0$ $\oplus$
Conclusion	f is dec.	f is inc.	f is dec.	f is inc.

Rel min   Rel max   Rel min



$f'(x)=\frac{2}{3}(x^2-9)^{-\frac{1}{3}}\cdot 2x$

$f'(x)=\frac{4x}{3(x^2-9)^{\frac{1}{3}}}$

Find Critical Numbers

$f'(x)=0$ $f'(x)=$ DNE Undefined

$\frac{4x}{3(x^2-9)^{1/3}}=0$ $x^2-9=0$

$4x=0$ $x^2=9$

$x=0$ $x=3$ or -3

TEST VALUE $x=-5,-1,1,5$

$f'(-5)=\frac{4(-5)}{3((-5)^2-9)^{\frac{1}{3}}}=\frac{-20}{3(25-9)^{\frac{1}{3}}}=\frac{-20}{3(16)^{\frac{1}{3}}}<0$

$f'(-1)=\frac{4(-1)}{3((-1)^2-9)^{1/3}}=\frac{-4}{3(-8)^{\frac{1}{3}}}>0$

$f'(1)=\frac{4(1)}{3((1)^2-9)^{\frac{1}{3}}}=\frac{4}{3(-8)^{\frac{1}{3}}}<0$

$f'(5)=\frac{4(5)}{3[(5)^2-9]^{\frac{1}{3}}}=\frac{20}{3\cdot(25-9)^{\frac{1}{3}}}=\frac{20}{3(16)^{\frac{1}{3}}}>0$

$g'(t) = \dfrac{(t^2)\cdot(4t^3) - (t^4+1)(2t)}{(t^2)^2}$

$g'(t) = \dfrac{4t^5 - 2t^5 - 2t}{t^4}$

$g'(t) = \dfrac{2t^5 - 2t}{t^4} = \dfrac{2t(t^4-1)}{t\cdot t^3}$

$g'(t) = \dfrac{2(t^4-1)}{t^3}$

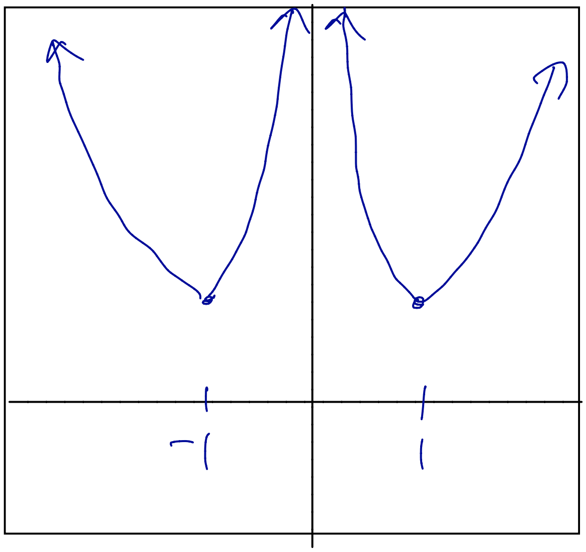
$g(t) = \dfrac{t^4}{t^2} + \dfrac{1}{t^2}$

$g(t) = t^2 + t^{-2}$

$g'(t) = 2t - 2t^{-3}$

Ex.4 Find the relative extrema of $g(t) = \dfrac{t^4+1}{t^2}$.

Interval	$(-\infty,-1)$	$(-1,0)$	$(0,1)$	$(1,\infty)$
Test Value	$t=-2$	$t=-\frac{1}{2}$	$t=\frac{1}{2}$	$t=2$
Sign of $g'(t)$	$g'(-2)<0$ $\ominus$	$g'(-\frac{1}{2})>0$ $\oplus$	$g'(\frac{1}{2})$ $\ominus$	$g'(2)>0$ $\oplus$
Conclusion	g is dec	g is inc	g is dec	g is inc



Find critical numbers:

Ⓐ $g'(t)=0$

$\dfrac{2(t^4-1)}{t^3}=0$

$2(t^4-1)=0$

$t^4-1=0$

$t^4=1$

$t=-\sqrt[4]{1}$, or $t=+\sqrt[4]{1}$

$t=-1$, or $t=1$

Ⓑ $g'(t)$ is undefined

$\dfrac{2(t^4-1)}{t^3}$ is undefined

$t^3=0$

$t=0$

$$g'(-2) = \frac{2((-2)^4-1)}{(-2)^3} = \frac{2(16-1)}{-8} = \frac{30}{-8} < 0$$

$$g'(2) = \frac{2(2^4-1)}{2^3} = \frac{2(16-1)}{8} = \frac{30}{8} > 0$$

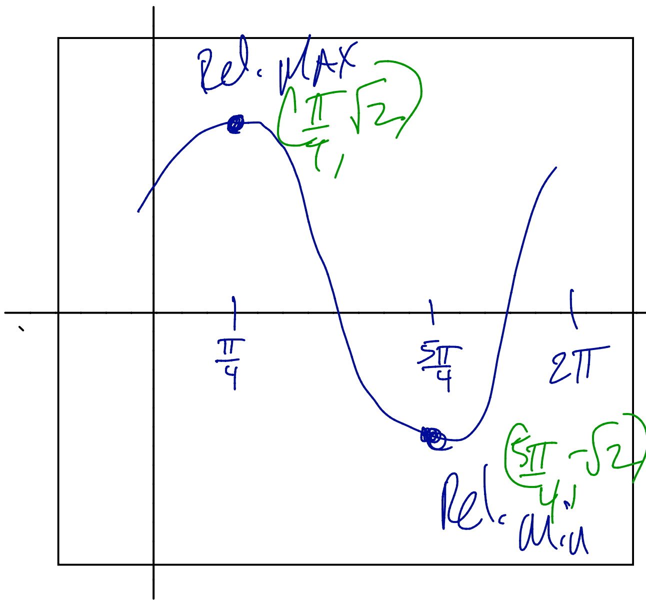
$$g'\left(-\tfrac{1}{2}\right) = \frac{2\left[\left(-\frac{1}{2}\right)^4-1\right]}{\left(-\frac{1}{2}\right)^3} = \frac{2\left(\frac{1}{16}-1\right)}{-\frac{1}{8}} = \frac{2\left(-\frac{15}{16}\right)}{-\frac{1}{8}} > 0$$

$$g'\left(\tfrac{1}{2}\right) = \frac{2\left[\left(\frac{1}{2}\right)^4-1\right]}{\left(\frac{1}{2}\right)^3} = \frac{2\left(-\frac{15}{16}\right)}{\frac{1}{8}} < 0$$

Ex.5 Find the relative extrema of $h(\theta)=\sin(\theta)+\cos(\theta)$ on the interval $(0,2\pi)$.

Rel MAX   Rel. Min

Interval	$(0, \frac{\pi}{4})$	$(\frac{\pi}{4}, \frac{5\pi}{4})$	$(\frac{5\pi}{4}, 2\pi)$
Test Value	$\theta = \frac{\pi}{6}$	$\theta = \pi$	$\theta = \frac{3\pi}{2}$
Sign of $h'(\theta)$	$h'(\frac{\pi}{6}) > 0$ $\oplus$	$h'(\pi) < 0$ $\ominus$	$h'(\frac{3\pi}{2}) > 0$ $\oplus$
Conclusion	h is increasing	h is decreasing	h is increasing



$$h'(\theta) = \frac{d}{d\theta}[\sin(\theta)+\cos(\theta)]$$

$$\boxed{h'(\theta) = \cos(\theta) - \sin(\theta)}$$

Find critical numbers:

(A) $h'(\theta) = 0$, or (B) $h'(\theta)$ is undefined — Never

$0 = \cos(\theta) - \sin(\theta)$

$\sin(\theta) = \cos(\theta)$

$\theta = \frac{\pi}{4}$ or $\theta = \frac{5\pi}{4}$

$$h'(\tfrac{\pi}{6}) = \cos(\tfrac{\pi}{6}) - \sin(\tfrac{\pi}{6}) = \frac{\sqrt{3}}{2} - \frac{1}{2} > 0$$

$$h'(\pi) = \cos(\pi) - \sin(\pi) = -1 - 0 = -1 < 0$$

$$h'(\tfrac{3\pi}{2}) = \cos(\tfrac{3\pi}{2}) - \sin(\tfrac{3\pi}{2}) = 0 - (-1) = 0 + 1 > 0$$

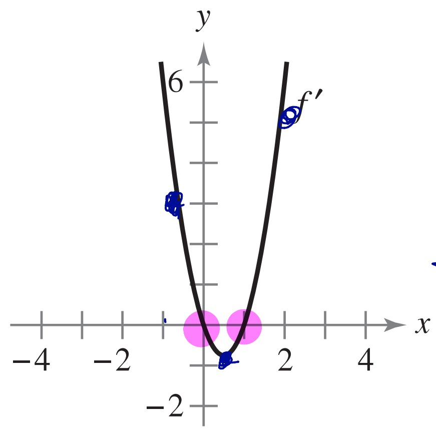
Find

$$h(\tfrac{\pi}{4}) = \sin(\tfrac{\pi}{4}) + \cos(\tfrac{\pi}{4}) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \frac{2\sqrt{2}}{2} = \sqrt{2}$$

$$h(\tfrac{5\pi}{4}) = \sin(\tfrac{5\pi}{4}) + \cos(\tfrac{5\pi}{4}) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = \frac{-2\sqrt{2}}{2} = -\sqrt{2}$$

Ex.6 Use the graph of f' to **(a)** identify the interval(s) on which f is increasing or decreasing, and **(b)** estimate the value(s) of x at which f has a relative maximum or minimum.

(i)

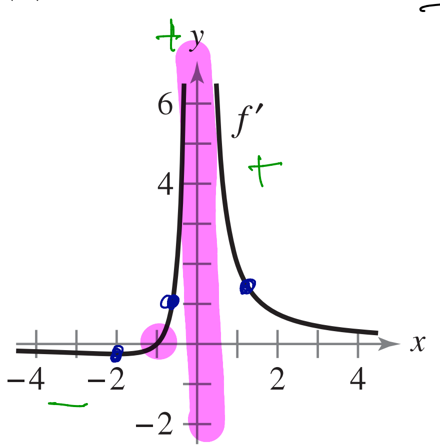


Interval:	$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
Test Value:	$x=-1$	$x=\frac{1}{2}$	$x=2$
Sign of $f'(x)$:	$f'(-1)>0$ ⊕	$f'(\frac{1}{2})<0$ ⊖	$f'(2)>0$ ⊕
Conclusion:	f is Inc	f is dec	f Inc

Rel MAX

Rel min

(ii)



Interval	$(-\infty, -1)$	$(-1, 0)$	$(0, \infty)$
Testvalue	$x=-2$	$x=-\frac{1}{2}$	$x=1$
Sign of $f'(x)$	$f'(-2)<0$ ⊖	$f'(-\frac{1}{2})>0$ ⊕	$f'(1)>0$ ⊕
Conclusion	f is dec	f is Inc	f is Inc

Rel min

Nothing